# THE 3-CLASS GROUPS OF $\mathbb{Q}(\sqrt[3]{p})$ AND ITS NORMAL CLOSURE 

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#### Abstract

We determine the 3-class groups of $\mathbb{Q}(\sqrt[3]{p})$ and $K=\mathbb{Q}(\sqrt[3]{p}, \sqrt{-3})$ when $p \equiv 4,7 \bmod 9$ is a prime and 3 is a cube modulo $p$. This confirms a conjecture made by Barrucand-Cohn, and proves the last remaining case of a conjecture of Lemmermeyer on the 3-class group of $K$.


## 1. Introduction

Let $p$ be a prime. Let $F=\mathbb{Q}(\sqrt[3]{p})$ and $K=\mathbb{Q}\left(\sqrt[3]{p}, \mu_{3}\right)$ the normal closure of $F$. Let $A_{F}$ (resp. $A_{K}$ ) be the 3 -class group (i.e., 3 -Sylow subgroup of the class group) of $F$ (resp. $K$ ). The paper aims to prove the following result.
Theorem 1.1. Assume that $p \equiv 4,7 \bmod 9$ is a prime such that the cubic residue symbol $\left(\frac{3}{p}\right)_{3}=1$. Then $A_{F} \cong \mathbb{Z} / 3 \mathbb{Z}$ and $A_{K} \cong(\mathbb{Z} / 3 \mathbb{Z})^{2}$.

This result confirms a conjecture made by Barrucand-Cohn in BaC70, §8], and later mentioned by Barrucand-Williams-Baniuk, Williams and Gerth in BWB76, §8, Conjecture 1], Wil82, p. 273] and [Ger05, p. 474]. Theorem 1.1] also completes a proof of a Lemmermeyer's conjecture on $A_{K}$ in Lem10, Conjecture 5, §1.10] when combining with the following known results:
(1) If $p \equiv 2 \bmod 3$, then the groups $A_{F}$ and $A_{K}$ are both trivial; see Hon71.
(2) If $p \equiv 1 \bmod 3$, then $A_{F}$ is cyclic non-trivial and $\operatorname{rk}\left(A_{K}\right)=1$ or 2 where $\operatorname{rk}\left(A_{K}\right)$ is the 3-rank of $A_{K}$; see Ger05.
(3) If $p \equiv 1 \bmod 9$, then $\operatorname{rk}\left(A_{K}\right)=2$ if and only if 9 divides $\left|A_{F}\right|$; see CaE05, Lemma 5.11] and Ger05.
(4) If $p \equiv 4,7 \bmod 9$ and $\left(\frac{3}{p}\right)_{3} \neq 1$; then $A_{F} \cong A_{K} \cong \mathbb{Z} / 3 \mathbb{Z}$; see BWB76 or Ger05.
We give two consequences of Theorem 1.1. Let $E_{K}$ be the group of units of $K$. Let $E_{K}^{\prime}$ be the subgroup of $E_{K}$ generated by the units of non-trivial subfields of $K$. Write $q=\left[E_{K}: E_{K^{\prime}}\right]$. One has ([BaC71, Theorem 12.1, 14.1])

$$
q=1 \text { or } 3 \quad \text { and } \quad h_{K}=\frac{q}{3} h_{F}^{2}
$$

Here $h_{K}\left(\right.$ resp. $\left.h_{F}\right)$ is the class number of $K($ resp. $F)$. Thus, if $p \equiv 4,7 \bmod 9$ and $\left(\frac{3}{p}\right)_{3}=1$, Theorem 1.1 implies that $q=3$. This confirms a conjecture made in ATIA20.

Assume $p \equiv 4,7 \bmod 9$. Theorem 1.1 implies that the norm equation $\mathbf{N}_{F / \mathbb{Q}}(x)=$ 3 has a solution $x \in \mathcal{O}_{F}$ if and only if $\left(\frac{3}{p}\right)_{3}=1$, as mentioned in Wil82, p. 273]. Since $\mathcal{O}_{F}=\mathbb{Z}[\sqrt[3]{p}]$, this is to say, the Diophantine equation

$$
x_{1}^{3}+p x_{2}^{3}+p^{2} x_{3}^{3}-3 p x_{1} x_{2} x_{3}=3
$$

has solutions $\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{Z}$ if and only if $\left(\frac{3}{p}\right)_{3}=1$.

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## 2. The proof

2.1. Chevalley's ambiguous class number formula. We first review the $S$ version of Chevalley's ambiguous class number formula which will be used. For a finite set $S$ of prime ideals of a number field $L$, the $S$-class group of $K$ is defined as

$$
\mathrm{Cl}_{K, S}:=\mathrm{Cl}_{K} /\langle[\mathfrak{p}]: \mathfrak{p} \in S\rangle,
$$

where $\mathrm{Cl}_{K}$ denotes the class group of $K$ and $[\mathfrak{p}]$ denotes the ideal class of $\mathfrak{p}$. Let $E_{K, S}:=\mathcal{O}_{K, S}^{\times}$denote the group of $S$-units of $K$. Let $L / K$ be a finite cyclic extension with Galois group $G$. For a finite set $S$ of prime ideals of $K$, we denote by $\mathrm{Cl}_{L, S}=\mathrm{Cl}_{L, S_{L}}$ for simplicity, where $S_{L}$ is the set of primes of $L$ lying above those in $S$. Chevalley's ambiguous class number formula states that the order of the $G$-invariant subgroup of $\mathrm{Cl}_{L, S}$ is given by

$$
\begin{equation*}
\left|\mathrm{Cl}_{L, S}^{G}\right|=\left|\mathrm{Cl}_{K, S}\right| \cdot \frac{\prod_{v \notin S} e_{v} \cdot \prod_{v \in S} e_{v} f_{v}}{[L: K] \cdot\left[E_{K, S}: E_{K, S} \cap \mathbf{N} L^{\times}\right]} \tag{2.1}
\end{equation*}
$$

Here the first product runs over all places of $K$ not in $S, e_{v}$ and $f_{v}$ are the ramification index and the residue degree of $v$ respectively, and $\mathbf{N}=\mathbf{N}_{L / K}$ is the norm map. For a proof of this formula, see [LiY20 for example. The unit index in (2.1) can be computed by Hilbert symbols provided that $L / K$ is a Kummer extension.

Proposition 2.1. Let $L / K$ be a cyclic extension of degree $d$ and $\mu_{d} \subset K$. Then $L=K(\sqrt[d]{a})$ for some $a \in K$. Let Ram be the set of ramified places of $K$. Define

$$
\begin{aligned}
\rho: \frac{E_{K, S}}{\left(E_{K, S}\right)^{d}} & \longrightarrow \prod_{v \in S \cup \operatorname{Ram}} \mu_{d} \\
x & \longmapsto\left(\left(\frac{x, a}{v}\right)_{d}\right)_{v \in S \cup \operatorname{Ram}} .
\end{aligned}
$$

Then the kernel of $\rho$ is given by

$$
\operatorname{Ker} \rho=\frac{E_{K, S} \cap \mathbf{N} L^{\times}}{\left(E_{K, S}\right)^{d}}
$$

and hence the size of the image is given by

$$
|\operatorname{Im}(\rho)|=\left[E_{K, S}: E_{K, S} \cap \mathbf{N} L^{\times}\right]
$$

which is at most $d^{|S \cup R a m|-1}$.
Proof. This result is a standard direct consequence of local class field theory, Hasse's norm theorem, and the product formula for Hilbert symbols. For details, see LOXZ20, §2].

If $\sigma \in \operatorname{Aut}(K)$ and $v$ is a prime of $K$, we have (loc. cit.)

$$
\begin{equation*}
\sigma\left(\frac{a, b}{v}\right)_{d}=\left(\frac{\sigma(a), \sigma(b)}{\sigma(v)}\right)_{d}, \quad a, b \in K^{\times} . \tag{2.2}
\end{equation*}
$$

For our applications, the degree $[L: K]$ is a power of a prime $\ell$. For any finite generated abelian group $A$, we denote by $A_{\ell}=A \otimes \mathbb{Z}_{\ell}$ where $\mathbb{Z}_{\ell}$ is the ring of $\ell$-adic integers. If $A$ is finite, $A_{\ell}$ is the $\ell$-primary subgroup of $A$. If there is no ambiguity, we write $a$ for $a \otimes 1 \in A_{\ell}$ for $a \in A$. Clearly, the formula (2.1) still holds by replacing $\left(\mathrm{Cl}_{L, S}\right)^{G}$ and $\mathrm{Cl}_{K, S}$ with $\left(\left(\mathrm{Cl}_{L, S}\right)_{\ell}\right)^{G}=\left(\mathrm{Cl}_{L, S}^{G}\right)_{\ell}$ and $\left(\mathrm{Cl}_{K, S}\right)_{\ell}$ respectively.

The following well known fact which is proved by counting the orbits of the $G$-action or by Nakayama's Lemma will be used frequently:

$$
\left(\mathrm{Cl}_{L, S}\right)_{\ell}=0 \text { if and only if }\left(\mathrm{Cl}_{L, S}^{G}\right)_{\ell}=0
$$

2.2. Proof of Theorem 1.1, From now on, assume $p \equiv 1 \bmod 3$. Denote by

- $k=\mathbb{Q}\left(\mu_{3}\right)$;
- $M^{+}$the unique cubic subfield of $\mathbb{Q}\left(\mu_{p}\right)$, which is real;
- $M=M\left(\mu_{3}\right)$ a quadratic extension of $M^{+}$;
- $L=K M=M\left(\sqrt[3]{p}, \mu_{3}\right)$;
- $A_{T}=\left(\mathrm{Cl}_{T}\right)_{3}$ for any number field $T$.


Proposition 2.2. Assume $p \equiv 1 \bmod 3$.
(1) There exists $\alpha \in \mathcal{O}_{k}$ such that $M=k(\sqrt[3]{p \alpha})$ and $p=\alpha \bar{\alpha}$;
(2) $A_{M}=0$ if and only if $p \equiv 4,7 \bmod 9$.

Proof. (1) Since $p \equiv 1 \bmod 3$, we can write $p=\alpha \bar{\alpha}$ for some $\alpha \in \mathcal{O}_{k}$. Note that ( $\alpha$ ) and $(\bar{\alpha})$ are exactly the ramified primes of $k$ in $M$. Now, since the class number of $k$ is 1 and $M / k$ is a Kummer extension, we have

$$
M=\mathbb{Q}(\sqrt[3]{\gamma}) \quad \text { and } \quad \gamma=\zeta_{3}^{a} \alpha^{b} \bar{\alpha}^{c}
$$

with $a, b, c \in\{0,1,2\}$ and $b c \neq 0$. Since $(3-a, 3-b, 3-c)$ gives the same field as $(a, b, c)$, we conclude that $M=k\left(\sqrt[3]{\zeta_{3}^{a} p}\right)$ or $k\left(\sqrt[3]{\zeta_{3}^{a} p \alpha}\right)$ for some $a=0,1,2$. Since $M$ is abelian over $\mathbb{Q}$ but $k\left(\sqrt[3]{\zeta_{3}^{a} p}\right) / \mathbb{Q}$ is not, $M$ must coincide with $k\left(\sqrt[3]{\zeta_{3}^{a} p \alpha}\right)$. By replacing $\alpha$ with $\zeta_{3}^{a} \alpha$, we have $M=k(\sqrt[3]{p \alpha})$ and $p=\alpha \bar{\alpha}$. This proves (1).
(2) We apply (2.1) and Proposition 2.1 to the cyclic cubic extension $M / k$. Let $\iota: k \hookrightarrow \mathbb{Q}_{p}$ be the embedding induced by $(\alpha)$. Then we have the following equalities of cubic Hilbert symbols:

$$
\begin{equation*}
\left(\frac{\zeta_{3}, p \alpha}{\alpha}\right)=\iota^{-1}\left(\frac{\iota\left(\zeta_{3}\right), p \iota(\alpha)}{\mathbb{Q}_{p}}\right)=\iota^{-1}\left(\frac{\iota\left(\zeta_{3}\right), \iota(\alpha)}{\mathbb{Q}_{p}}\right)^{-1}=\zeta_{3}^{(p-1) / 3} \tag{2.3}
\end{equation*}
$$

Hence this symbol as well as the index $\left[E_{k}: E_{k} \cap \mathbf{N} M^{\times}\right]$is trivial if and only if $p \equiv 1 \bmod 9$. Thus

$$
\left|A_{M}^{G}\right|=\frac{3^{2}}{3 \cdot\left[E_{k}: E_{k} \cap \mathbf{N} M^{\times}\right]}=1
$$

if and only if $p \equiv 4,7 \bmod 9$. By Nakayama's lemma, it turns out that $A_{M}$ is trivial if and only if $p \equiv 4,7 \bmod 9$. This completes the proof of Proposition 2.2.

Let $\mathfrak{p}$ (resp. $\mathfrak{p}^{\prime}$ ) be the unique prime of $M$ (resp. $K$ ) lying above $\alpha \mathcal{O}_{k}$. Then $\alpha \mathcal{O}_{M}=\mathfrak{p}^{3}$ and $\alpha \mathcal{O}_{K}=\mathfrak{p}^{\prime 3}$.
Proposition 2.3. Assume that $p \equiv 1 \bmod 3$ and $\left(\frac{3}{p}\right)_{3}=1$.
(1) The extensions $L / K$ and $F M^{+} / F$ are both abelian unramified cubic extensions.
(2) The primes $\mathfrak{p}$ and $\mathfrak{p}^{\prime}$ both split in $L$.

Proof. (1) Since $L=K M^{+}$, we have that $L / K$ is unramified outside the primes above $p$. Denote by $I_{(\alpha)}$ the inertia group of $(\alpha)=\alpha \mathcal{O}_{k}$ in the abelian extension $L / k$. By local class field theory and noting that the completion of $k$ at $(\alpha)$ is $\mathbb{Q}_{p}$, we have a surjection

$$
\mathbb{Z}_{p}^{\times} \rightarrow I_{(\alpha)}
$$

It follows that $I_{(\alpha)}$ can not be $\operatorname{Gal}(L / k) \cong(\mathbb{Z} / 3 \mathbb{Z})^{2}$. On the other hand, $I_{(\alpha)}$ is non-trivial since $(\alpha)$ is ramified in $K$ and $M$. This shows that $\mathfrak{p}$ and $\mathfrak{p}^{\prime}$ must be unramified in $L$. An entirely similar argument for the prime $(\bar{\alpha})=\bar{\alpha} \mathcal{O}_{k}$ shows that $L / M$ and $L / K$ are both unramified outside the primes above $\bar{\alpha}$. This shows that $L / K$ is unramified everywhere.

For the extension $F M^{+} / F$, first note that it is unramified outside $\sqrt[3]{p} \mathcal{O}_{F}$ as $M^{+} / \mathbb{Q}$ is unramified outside $p$. We claim that $\sqrt[3]{p} \mathcal{O}_{F}$ is also unramified in $F M^{+}$. Otherwise, since $K / F$ is unramified at $\sqrt[3]{p} \mathcal{O}_{F}$, the prime of $K$ above $\sqrt[3]{p}$ would be ramified in $L$. But this contradicts that $L / K$ is unramified whence the claim holds. This proves (1).
(2) We have just shown that $F M^{+}$is contained in the Hilbert class field of $F$. By class field theory, the principal prime $\sqrt[3]{p} \mathcal{O}_{F}$ splits in $F M^{+}$. It follows that $\mathfrak{p}^{\prime}$ and $\mathfrak{p}$ both split in $L$.

Lemma 2.4. (1) If $p \equiv 4,7 \bmod 9$, then 3 is totally ramified in $K$.
(2) If $p \equiv 1 \bmod 3$ and 3 is a cube modulo $p$, then $\left(1-\zeta_{3}\right) \mathcal{O}_{k}$ splits in $M$.

Proof. (1) Since $(x+p)^{3}-p$ is an Eisenstein polynomial, 3 is totally ramified in $F$. Since 3 is also ramified in $k$, it follows that 3 is totally ramified in $K$ by counting the ramification degrees.
(2) Fix the canonical isomorphism

$$
\begin{aligned}
(\mathbb{Z} / p \mathbb{Z})^{\times} & \cong \operatorname{Gal}\left(\mathbb{Q}\left(\mu_{p}\right) / \mathbb{Q}\right) \\
a & \mapsto\left(\sigma_{a}: \zeta_{p} \mapsto \zeta_{p}^{a}\right) .
\end{aligned}
$$

By definition, $M^{+}$is the subfield of $\mathbb{Q}\left(\mu_{p}\right)$ fixed by $(\mathbb{Z} / p \mathbb{Z})^{\times 3}$. Our assumptions imply that $\sigma_{3}$ is trivial on $M^{+}$whence 3 splits in $M^{+}$. It follows that $\left(1-\zeta_{3}\right) \mathcal{O}_{k}$ must split in $M$.

We need the following elementary fact on the local field $\mathbb{Q}_{3}\left(\mu_{3}\right)$.
Lemma 2.5. If $a, b \in \mathbb{Z}$ with $3 \nmid a b$, then the cubic Hilbert symbol of $a$ and $b$ in $\mathbb{Q}_{3}\left(\mu_{3}\right)$ is trivial.

Proof. By convergence of the Taylor expansion of $(1+9 x)^{1 / 3}$ on $\mathbb{Z}_{3}\left[\mu_{3}\right]$, every element in $1+9 \mathbb{Z}_{3}\left[\mu_{3}\right]$ is a cube. Note that -1 is a cube whence the cubic Hilbert symbol $\left(\frac{a, a}{\mathbb{Q}_{3}\left(\mu_{3}\right)}\right)=1$. Thus, we only need to show the triviality of the symbol

$$
\left(\frac{4,7}{\mathbb{Q}_{3}\left(\zeta_{3}\right)}\right)=\left(\frac{4,2}{\mathbb{Q}_{3}\left(\zeta_{3}\right)}\right)=\left(\frac{2,2}{\mathbb{Q}_{3}\left(\zeta_{3}\right)}\right)^{2}=1
$$

Theorem 2.6. Assume that $p \equiv 4,7 \bmod 9$ and $\left(\frac{3}{p}\right)_{3}=1$. Then $A_{L}$ is non-trivial and 3 does not divides $\left|\mathrm{Cl}_{L,\{\mathfrak{p}\}}\right|$.

Proof. We first apply (2.1) on $L / M$ with $S=\emptyset$ to prove 3 divides $\left|A_{L}^{G}\right|$ where $G=\operatorname{Gal}(L / M)$. By Proposition 2.3 and Lemma 2.4, exactly the three primes $\mathfrak{l}, \sigma(\mathfrak{l}), \sigma^{2}(\mathfrak{l})$ of $M$ lying above $\left(1-\zeta_{3}\right) \mathcal{O}_{k}$ are ramified in $L / M$, where $\sigma$ is a generator of $\operatorname{Gal}(M / k)$. By Proposition 2.2, we know that $\left|A_{M}\right|=1$. It remains to compute
the unit index. Note that $L=M(\sqrt[3]{p})$. To apply Proposition 2.1 we define

$$
\begin{aligned}
\rho: E_{M} & \longrightarrow \mu_{3}^{3} \\
u & \longmapsto\left(\left(\frac{u, p}{\mathfrak{l}}\right),\left(\frac{u, p}{\sigma(\mathfrak{l})}\right),\left(\frac{u, p}{\sigma^{2}(\mathfrak{l})}\right)\right) .
\end{aligned}
$$

Since $M / M^{+}$is a CM-extension, the group $\left(E_{M}\right)_{3}$ is generated by $\left(E_{M^{+}}\right)_{3}$ and $\zeta_{3}$ by Was82, Theorem 4.12]. The completion of $M^{+}$at a prime above 3 is $\mathbb{Q}_{3}$. It follows that $\eta \equiv a \bmod 9$ with $a \in \mathbb{Z}$ for any $\eta \in E_{M^{+}}$. Then by Lemma 2.5,

$$
\begin{equation*}
\left|\rho\left(E_{M^{+}}\right)\right|=1 \tag{2.4}
\end{equation*}
$$

Now we compute $\rho\left(\zeta_{3}\right)$. Since $\sigma\left(\zeta_{3}\right)=\zeta_{3}$, by (2.2) we have

$$
\left(\frac{\zeta_{3}, p}{\mathfrak{l}}\right)=\left(\frac{\zeta_{3}, p}{\sigma(\mathfrak{l})}\right)=\left(\frac{\zeta_{3}, p}{\sigma^{2}(\mathfrak{l})}\right) .
$$

By Lemma 2.4, the completion of $M$ at $\mathfrak{l}$ is $\mathbb{Q}_{3}\left(\mu_{3}\right)$. Applying the product formula for cubic Hilbert symbols on the field $\mathbb{Q}\left(\mu_{3}\right)$ gives

$$
\left(\frac{\zeta_{3}, p}{(\alpha)}\right)\left(\frac{\zeta_{3}, p}{(\bar{\alpha})}\right)\left(\frac{\zeta_{3}, p}{\left(1-\zeta_{3}\right)}\right)=1 .
$$

By (2.3) and our assumption $p \equiv 4,7 \bmod 9$, we obtain that

$$
\begin{equation*}
\left(\frac{\zeta_{3}, p}{\left(1-\zeta_{3}\right)}\right) \neq 1 \text { and }\left(\frac{\zeta_{3}, p}{\mathbb{Q}_{3}\left(\mu_{3}\right)}\right) \neq 1 \tag{2.5}
\end{equation*}
$$

This proves that $\rho\left(\zeta_{3}\right)=\zeta_{3}^{ \pm 1}(1,1,1)$. Combining with (2.4), we conclude that $\left|\rho\left(E_{M}\right)_{3}\right|=3$. Then Chevalley's formula gives

$$
\left|A_{L}^{G}\right|=\frac{3^{3}}{3 \times 3}=3
$$

In particular, $\left|A_{L}\right| \geq 3$.
Next, we apply Chevalley's formula on $L / M$ with $S=\{\mathfrak{p}\}$ to compute $\mathrm{Cl}_{L,\{\mathfrak{p}\}}^{G}$. Define

$$
\beta=\frac{\sqrt[3]{p \alpha}}{\mathbf{N}_{\mathbb{Q}\left(\mu_{p}\right) / M^{+}}\left(1-\zeta_{p}\right)}
$$

Note that $\beta^{3}$ generates the ideal $\alpha \mathcal{O}_{M}$ whence $\beta \mathcal{O}_{M}=\mathfrak{p}$. It follows that $\left(E_{M,\{\mathfrak{p}\}}\right)_{3}$ is generated by $\beta, \zeta_{3}$ and $E_{M^{+}}$. We claim that

$$
\left(\frac{\beta, p}{\mathfrak{l}}\right) \neq\left(\frac{\beta, p}{\sigma(\mathfrak{l})}\right) .
$$

Indeed, by (2.2), the right hand side equals the Hilbert symbol of $\sigma^{-1}(\beta)$ and $p$ at $\mathfrak{l}$. Note that $\sigma^{-1}(\beta)=\zeta_{3}^{ \pm 1} \beta \eta$ for some $\eta \in E_{M^{+}}$. Thus the inequality follows from (2.4) and (2.5). By Proposition 2.1, this shows that the index

$$
\left[E_{M,\{\mathfrak{p}\}}: E_{M,\{\mathfrak{p}\}} \cap \mathbf{N} L^{\times}\right]=9
$$

By Proposition 2.3 the prime $\mathfrak{p}$ splits in $L$. It follows from (2.1) that 3 does not divide $\left|\mathrm{Cl}_{L,\{\mathfrak{p}\}}^{G}\right|$ whence 3 does not divide $\left|\mathrm{Cl}_{L,\{\mathfrak{p}\}}\right|$ by Nakayama's Lemma. This completes the proof.

Proof of Theorem 1.1, By Theorem [2.6, $A_{L}$ is non-trivial. It follows that, by Nakayama's lemma, we have $\left|A_{L}^{\operatorname{Gal}(L / K)}\right| \geq 3$. Since $L / K$ is unramified everywhere, by Hasse's norm theorem and local class field theory (or Proposition 2.3), we have the unit index $\left[E_{K}: E_{K} \cap \mathbf{N}\left(L^{\times}\right)\right]=1$. Then applying Chevalley's formula with $S=\emptyset$ to the extension $L / K$ gives

$$
\left|A_{K}\right| \geq 9
$$

Recall that $\mathfrak{p}^{\prime}$ is the prime of $K$ lying above $\alpha \mathcal{O}_{k}$. Note that $\mathfrak{p}^{\prime} \mathcal{O}_{L}=\mathfrak{p} \mathcal{O}_{L}$, we have $\mathrm{Cl}_{L,\left\{\mathfrak{p}^{\prime}\right\}}=\mathrm{Cl}_{L,\{\mathfrak{p}\}}$ by definition. Since 3 does not divide $\left|\mathrm{Cl}_{L,\left\{\mathfrak{p}^{\prime}\right\}}\right|$ by Theorem 2.6 and $\mathfrak{p}^{\prime}$ splits in $L$ by Proposition 2.3. Chevalley's formula with $S=\left\{\mathfrak{p}^{\prime}\right\}$ will imply that $\left(\mathrm{Cl}_{K,\left\{\mathfrak{p}^{\prime}\right\}}\right)_{3} \cong \mathbb{Z} / 3 \mathbb{Z}$ if we can show that

$$
\left[E_{K,\left\{\mathfrak{p}^{\prime}\right\}}: E_{K,\left\{\mathfrak{p}^{\prime}\right\}} \cap \mathbf{N}\left(L^{\times}\right)\right]=1
$$

Because $\mathfrak{p}^{\prime}$ splits in $L / K$ by Proposition 2.3, the local extension at $\mathfrak{p}^{\prime}$ is trivial. Thus any $\left\{\mathfrak{p}^{\prime}\right\}$-unit is a local norm at $\mathfrak{p}^{\prime}$ whence is a local norm at every place of $K$ as $L / K$ is unramified. By Hasse's norm theorem, the above unit index is indeed trivial.

The equality $\mathfrak{p}^{\prime 3}=\alpha \mathcal{O}_{K}$ implies that $\left|\mathrm{Cl}_{K}\right| \leq 3\left|\mathrm{Cl}_{K,\left\{\mathfrak{p}^{\prime}\right\}}\right|$. It follows that

$$
\left|A_{K}\right| \leq 9
$$

Hence $\left|A_{K}\right|=9$ and then $\left|A_{F}\right|=3$ by Hon71, Lemma 1].
Let $\tau$ be the non-trivial element of $\Delta=\operatorname{Gal}(K / F)$. Since $\Delta$ is of order 2, we have a decomposition of $\mathbb{Z}_{3}[\Delta]$-modules

$$
A_{K}=A_{K}^{+} \oplus A_{K}^{-}, \text {where } A_{K}^{ \pm}=\left\{a \in A_{K} \mid \tau(a)=a^{ \pm 1}\right\}
$$

It is well known that $\left|A_{K}^{+}\right|=\left|A_{F}\right|=3$ (for example, using (2.1)). Thus $A_{K}$ has a direct factor $\mathbb{Z} / 3 \mathbb{Z}$. This implies that $A_{K} \cong(\mathbb{Z} / 3 \mathbb{Z})^{2}$, completing the proof of Theorem 1.1 .

René Schoof informed us that, using Theorem 1.1 one can go back to improve Theorem 2.6. Under the assumptions of Theorem [2.6, we indeed have

$$
\begin{equation*}
A_{L} \cong \mathbb{Z} / 3 \mathbb{Z} \tag{2.6}
\end{equation*}
$$

We write his argument as follows. In what follows, let $G=\operatorname{Gal}(L / k)$ so that $G \cong(\mathbb{Z} / 3 \mathbb{Z})^{2}$.
Lemma 2.7. For any subgroup $H$ of $G$ of order 3 , we have $\left|A_{L}^{H}\right|=3$.
Proof. The nontrivial intermediate fields of $L / k$ are $K, M, k(\sqrt[3]{\alpha}), k(\sqrt[3]{\bar{\alpha}})$, where $\alpha$ is as in Proposition [2.2. The case $H=\operatorname{Gal}(L / M)$ has been proved in the proof of Theorem [2.6. For $H=\operatorname{Gal}(L / K)$, as in proof of Theorem [1.1, one has $\left|A_{L}^{H}\right|=\frac{\left|A_{K}\right|}{3}=3$; here the last equality is by Theorem 1.1.

Now consider the case $H=\operatorname{Gal}(L / k(\sqrt[3]{\alpha}))$. We first prove $A_{k(\sqrt[3]{\alpha})}=0$ by the same method as in Proposition [2.2. There are precisely two primes $(\alpha),\left(1-\zeta_{3}\right)$ of $k$ ramified in $k(\sqrt[3]{\alpha}) / k$. Since $p \equiv 4,7 \bmod 9$, we have $\left(\frac{\zeta_{3}, \alpha}{(\alpha)}\right) \neq 1$ as in (2.3). Then Chevalley's formula for $k(\sqrt[3]{\alpha}) / k$ and Nakayama's lemma give $A_{k(\sqrt[3]{\alpha})}=0$. Now there are precisely three primes of $k(\sqrt[3]{\alpha})$ ramified in $L / k(\sqrt[3]{\alpha})$ which are the primes lying above $\bar{\alpha}$. In particular, the completion at such a prime $\mathfrak{P}$ is isomorphic to $\mathbb{Q}_{p}$. Note that $L=k(\sqrt[3]{\alpha})(\sqrt[3]{p})$. Hence $\left(\frac{\zeta_{3}, p}{\mathfrak{P}}\right) \neq 1$ as $p \equiv 4,7 \bmod 9$. Applying Chevalley's formula to $L / k(\sqrt[3]{\alpha})$ gives $\left|A_{L}^{H}\right| \leq 3$ whence it is 3 by Nakayama's lemma. The case $H=\operatorname{Gal}(L / k(\sqrt[3]{\alpha}))$ can be proved in an entirely similar way.

Lemma 2.8 (Schoof). Let $R$ be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$. Suppose that $\operatorname{dim} \mathfrak{m} / \mathfrak{m}^{2}>1$. Let $J \subset \mathfrak{m}$ be an ideal for which $J+(x)=\mathfrak{m}$ for every $x \in \mathfrak{m}-\mathfrak{m}^{2}$. Then $J=\mathfrak{m}$.

Proof. We may replace $J$ by $J+\mathfrak{m}^{2}$, because if $J+\mathfrak{m}^{2}$ is equal to $\mathfrak{m}$, then also $J=\mathfrak{m}$ by Nakayama's lemma. So we have $\mathfrak{m}^{2} \subset J$. Then $W=J / \mathfrak{m}^{2}$ is a sub vector space of $V=\mathfrak{m} / \mathfrak{m}^{2}$ and we know that $W+v=V$ for every non-zero vector $v \in V$. Since $\operatorname{dim} V>1, W$ cannot be zero. So we can find a non-zero $v \in W$. It follows that $W=W+v=V$ and we are done.

Proof of (2.6). Clearly, $A_{L}$ is a module over the complete local ring $R=\mathbb{Z}_{3}[G]$. The maximal ideal $\mathfrak{m}$ of $R$ is generated by the elements $s-1$ with $s \in G$. By Theorem [2.6. $A_{L}$ is cyclic over the ring $R$. So, $A_{L}=R / J$ for some ideal $J \subset R$. Let $s \in G$ be a nontrivial element and write $H$ for the subgroup generated by $s$. We have an exact sequence

$$
0 \rightarrow A_{L}^{H} \rightarrow A_{L} \xrightarrow{s-1} A_{L} \rightarrow A_{L} /(s-1) A_{L} \rightarrow 0 .
$$

The rightmost term has order 3 for every nontrivial $s \in G$ by Lemma 2.7. In other words, $J+(s-1)=\mathfrak{m}$ for every nontrivial $s \in G$. Since $\mathfrak{m} / \mathfrak{m}^{2}$ is generated by the elements $s-1(s \in G)$ as a $R / \mathfrak{m}$-vector space, it follows from Lemma 2.8 that $J=\mathfrak{m}$. Hence $\left|A_{L}\right|=3$.

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